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Inertia and Controllability in Infinite Dimensions

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The *inertia* of an $n \times n$ matrix A with complex entries is the integer triple $\operatorname{In}(A) = (\pi(A), \nu(A), \delta(A))$, where $\pi(A)$ is the number of eigenvalues of A (counted according to algebraic multiplicity) in the open right half-plane, $\nu(A)$ is the number of eigenvalues in the open left half-plane, and $\delta(A)$ is the number of eigenvalues on the imaginary axis. Two classical inertia theorems are Sylvester's theorem [8, I, p. 334] and Lyapunov's theorem [8, II, p. 189]. Sylvester's theorem says that if S is a nonsingular $n \times n$ matrix and H is an Hermitian $n \times n$ matrix, then $\operatorname{In}(H) = \operatorname{In}(S^*HS)$. Lyapunov's theorem says that, for a given $n \times n$ matrix A, there is a positive invertible matrix H such that $AH + HA^*$ is positive and invertible if and only if $\operatorname{In}(A) = (n, 0, 0)$.

Taussky [17] and Ostrowski and Schneider [12] proved the following generalization, which is often called the main inertia theorem.

MAIN INERTIA THEOREM. *If A and H are $n \times n$ complex matrices, H is Hermitian, and $AH + HA^*$ is positive and invertible, then $\delta(A) = \delta(H) = 0$ and* $\operatorname{In}(A) = \operatorname{In}(H)$.

A pair of $n \times n$ matrices (A, W) is called *controllable* if the $n \times n^2$ matrix

$$(W, AW, A^2W, ..., A^{n-1}W)$$

has rank n. If W is invertible, then obviously any pair (A, W) is controllable. Chen [6] and Wimmer [19] generalized the inertia theorem as follows:

CHEN–WIMMER THEOREM. *If A, H, and W are $n \times n$ complex matrices, $H = H^*$, W is positive, (A, W) is controllable, and $AH + HA^* = W$, then $\delta(A) = \delta(H) = 0$ and* $\operatorname{In}(A) = \operatorname{In}(H)$.

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In this paper we formulate and prove versions of these theorems for bounded linear operators acting on a complex Hilbert space $\mathscr{H}$. We denote by $B(\mathscr{H})$ the set of bounded linear operators on $\mathscr{H}$. Several authors have previously considered infinite-dimensional versions of these theorems, for instance [18, 3, 4, 16, 5]. We will indicate below how our results overlap with the known results. One of our goals, besides generalizing these results to infinite dimensions, is to provide "structural" proofs of the results.

In order to formulate and prove our results, we need the notion of the matrix (or operator) sign function. For $A \in B(\mathscr{H})$, we say $\delta(A)=0$ if $A-tiI$ is invertible in $B(\mathscr{H})$ for every real number t. If $\delta(A)=0$, then the spectrum of A, $\sigma(A)$, splits into two parts, one on each side of the imaginary axis. Let $S_\pm$ be Cauchy domains with boundaries $C_\pm$ such that $\{z: z \in \sigma(A), \pm\mathrm{Re}(z)>0\} \subset S_\pm$ and $\pm\mathrm{Re}(z)>\varepsilon$ for all $z \in S_\pm$, where $\varepsilon>0$. Let

$$s^\pm(A)=(1/2\pi i)\int_{C_\pm}(zI-A)^{-1}\,dz.$$

Then $s^\pm(A)^2=s^\pm(A)$, and $s^+(A)+s^-(A)=I$. See [13, Section 2.1] for details concerning this construction. Let $s(A)=s^+(A)-s^-(A)$. Note that $s(A)^2=I$. In finite dimensions, the dimension of $s^+(A)(\mathscr{H})$ is $\pi(A)$ and the dimension of $s^-(A)(\mathscr{H})$ is $\nu(A)$. In infinite dimensions, if $\delta(A)=\delta(H)=0$ we will say that A and H have the same inertia if there are bijective bounded linear operators from $s^+(A)(\mathscr{H})$ to $s^+(H)(\mathscr{H})$ and from $s^-(A)(\mathscr{H})$ to $s^-(H)(\mathscr{H})$. In all cases that we say A and H have the same inertia, we actually identify (or at least could identify) the bijective bounded linear maps involved.

We will use the following iterative algorithm to compute $s(A)$. This algorithm is due to Roberts [14, Sections 1.2 and 1.3]. The argument given by Roberts is stated only for matrices, but the proof is valid for any $A \in B(\mathscr{H})$ with $\delta(A)=0$. Let $A_0=A$,

$$A_{r+1}=1/2(A_r+A_r^{-1}), \qquad r=0, 1, \ldots.$$

Then the sequence A_r converges to $s(A)$ in norm.

Let (A, W) be a pair of operators in $B(\mathscr{H})$. The pair (A, W) is called *weakly controllable* if $\mathscr{H}$ is the norm closed linear span of the set $\{A^jWx: j \geqslant 0, x \in \mathscr{H}\}$, and *strongly controllable* if $\mathscr{H}$ is the linear span of this set; see [7]. If (A, W) is strongly controllable, then there is an n and $C_k \in B(\mathscr{H})$, $k=0, 1, \ldots, n$, such that

$$I=\sum_{k=0}^{n} A^k W C_k,$$

see [7, p. 79]. (The proof in [7] is valid in our context, operators on Hilbert space, although it is not valid for operators on Banach spaces. See the introduction of [15] for a discussion of this.) In finite dimensions the following theorem is in [6, Theorems 1 and 2] and [19, Theorem 2].

THEOREM 1. *Let A, H, and W be in $B(\mathscr{H})$ with H hermitian and W positive. If $AH + HA^* = W$ and (A, W) is strongly controllable, then $\delta(A) = 0 = \delta(H)$.*

Proof. An easy calculation shows that for every nonnegative integer k, real t, and x_m in $\mathscr{H}$,

$$2 \operatorname{Re}((A + tiI) A^k H(A^k)^*) = A^k W(A^k)^*;$$

so that

$$\|\sqrt{W}(A^k)^* x_m\|^2 = 2 \operatorname{Re}(A^k H(A^k)^* x_m, (A^* - tiI) x_m). \qquad (*)$$

If ti were in the boundary of $\sigma(A^*)$, there would exist unit vectors x_m such that $(A^* - tiI) x_m$ converges to zero in norm. Hence, by $(*)$, $\sqrt{W}(A^k)^* x_m$ converges to zero in norm. So, for each n, $\sum_{k=0}^{n} (C_k)^* W(A^k)^*$ is singular, which contradicts (A, W) being strongly controllable. Hence $\delta(A^*) = 0$.

If $\delta(H) \neq 0$, there exist unit vectors x_m such that $H(A^0)^* x_m = Hx_m$ converges to zero in norm. Assume, as inductive assumption, that $H(A^k)^* x_m$ converges to zero in norm as m goes to infinity. Then, by $(*)$, $W(A^k)^* x_m$ converges to zero, so

$$H(A^{k+1})^* x_m = W(A^k)^* x_m - AH(A^*)^k x_m$$

converges to zero. Thus $H(A^k)^* x_m$ and, by $(*)$, $W(A^k)^* x_m$ converge to zero as m goes to infinity for each $k = 0, 1, 2, \ldots$. As in the first part of the proof, this contradicts (A, W) being strongly controllable. Hence $\delta(H) = 0$. Q.E.D.

LEMMA 2. *Let A, H, $W \in B(\mathscr{H})$ with $AH + HA^* = W$, W positive, and $\delta(A) = 0$. Set $s^{\pm} = s^{\pm}(A)$. Then*

$$s^{\pm} H(s^{\pm})^* = \int_{\mp\infty}^{0} e^{tA} s^{\pm} W(s^{\pm})^* e^{tA^*} \, dt,$$

where the improper integrals are norm convergent.

Proof. Let $S_{\pm}$ be Cauchy domains with boundaries $C_{\pm}$ such that

$$\{z \in \sigma(A) \colon \pm \operatorname{Re}(z) > 0\} \subset S_{\pm} \subset \{z \colon \pm \operatorname{Re}(z) > \varepsilon\},$$

where $\varepsilon > 0$. By [13, Section 2.1],

$$e^{At}s^{\pm} = (1/2\pi i)\int_{C_{\pm}} e^{zt}(zI - A)^{-1}\, dz.$$

Let M be a number such that

$$\|(zI - A)^{-1}\| \leqslant M \qquad \text{for all } z \text{ in } C_{+} \cup C_{-}.$$

Then for $z \in C_{\pm}$, $\|e^{zt}(zI - A)^{-1}\| \leqslant Me^{\pm\varepsilon t}$ for all $\mp t > 0$. It follows that for $\mp t > 0$

$$\|e^{At}s^{\pm}\| \leqslant (1/2\pi)\, MLe^{\pm\varepsilon t},$$

where L is the length of $C_{+} \cup C_{-}$. It follows that the improper integrals in the lemma are norm convergent. Now

$$\frac{d}{dt}(s^{\pm}e^{tA}He^{tA^*}(s^{\pm})^*) = s^{\pm}e^{tA}(AH + HA^*)\, e^{tA^*}(s^{\pm})^*$$
$$= s^{\pm}e^{tA}We^{tA^*}(s^{\pm})^*,$$

so

$$\int_{\mp\infty}^{0} e^{tA}s^{\pm}W(s^{\pm})^*\, e^{tA^*}\, dt = s^{\pm}H(s^{\pm})^*. \qquad \text{Q.E.D.}$$

The next theorem is the only really new ingredient in our approach to the inertia theorems.

THEOREM 3. *Let A, H, $W \in B(\mathscr{H})$ with $AH + HA^* = W$, W positive, and $\delta(A) = 0$. Then $s(A)H + Hs(A)^* = L$ is positive and $\ker(L) \subset \bigcap_{j=0}^{\infty} \ker(WA^{*j})$.*

Proof. Let $s = s(A)$, $s^{\pm} = s^{\pm}(A)$. Let $A_0 = A$ and $A_{r+1} = \frac{1}{2}(A_r + A_r^{-1})$. We claim that $A_rH + HA_r^* = W_r$, where $W_0 = W$ and $W_{r+1} = \frac{1}{2}(W_r + A_r^{-1}W_rA_r^{*-1})$. This is trivially true for $r = 0$. If $\delta(A) = 0$, then $\delta(A_r) = 0$ by the spectral mapping theorem and by [14] the sequence A_r converges in norm to s. If $A_rH + HA_r^* = W_r$, then $HA_r^{-1*} + A_r^{-1}H = A_r^{-1}WA_r^{-1*}$ so that $A_{r+1}H + HA_{r+1}^* = \frac{1}{2}(W_r + A_r^{-1}WA_r^{-1*})$, and the claim is proved. Taking limits in $A_rH + HA_r^* = W_r$, we see that $sH + Hs^* = L$, where L is the limit of the sequence W_r. Since each W_r is positive, L is positive. Since $s^2 = I$, we see that $sLs^* = Hs^* + sH = L$. It follows that

$$s^{+}L(s^{-})^* + s^{-}L(s^{+})^* = -s^{+}L(s^{-})^* - s^{-}L(s^{+})^*$$

so that

$$s^{+}L(s^{-})^* + s^{-}L(s^{+})^* = 0$$

and

$$L = s^+ L(s^+)^* + s^- L(s^-)^*.$$

Since $sH + Hs^* = L$, we have

$$\begin{aligned} s^+ L(s^+)^* &= s^+ H(s^+)^* + s^+ H(s^+)^* \\ &= 2s^+ H(s^+)^*. \end{aligned}$$

Likewise, $s^- L(s^-)^* = -2s^- H(s^-)^*$. If $Lx = 0$, then since L is positive, $s^\pm L s^{\pm *} x = 0$. Hence $s^\pm H s^{\pm *} x = 0$. It follows from Lemma 2 that $We^{A^* t} s^{+*} x = 0$ for all $t \leqslant 0$ and $We^{A^* t} s^{-*} x = 0$ for all $t \geqslant 0$. By differentiating, we see that $WA^{*j} s^{\pm *} x = 0$ for all $j = 0, 1, 2, \ldots$. Since $s^+ + s^- = I$, this implies that $WA^{*j} x = 0$ for all j. Q.E.D.

We note that if $\mathscr{H}$ is finite-dimensional and (A, W) is controllable, then the L in Theorem 3 is invertible. Theorem 1 and the main inertia theorem then imply that H and $s(A)$ have the same inertia, so H and A have the same inertia. So in finite dimensions Theorem 3 reduces the Chen–Wimmer theorem to the main inertia theorem. The following theorem implies the Chen–Wimmer theorem in the finite-dimensional case without using the main inertia theorem. In the case that W is invertible, Cain proved in [3, Theorem 5] that the pairs $(s^+(A)(\mathscr{H}), s^+(H)(\mathscr{H}))$ and $(s^-(A)(\mathscr{H}), s^-(H)(\mathscr{H}))$ both have the same Hilbert space dimension.

THEOREM 4. *Let A, H, and W be in $B(\mathscr{H})$ with H Hermitian and W positive. If $AH + HA^* = W$ and (A, W) is strongly controllable, then $\delta(A) = \delta(H) = 0$ and the linear transformations*

$$s^\pm(H)\colon s^\pm(A)(\mathscr{H}) \to s^\pm(H)(\mathscr{H})$$

are both one-to-one and onto.

Proof. Theorem 1 stated that $\delta(A) = \delta(H) = 0$. By Theorem 3 $s(A)H + Hs(A)^* = L$ is positive. If L were not invertible, there would exist a sequence of unit vectors $x_n \in \mathscr{H}$ such that Lx_n converges to zero in norm. For any generalized limit Lim (cf. [21, p. 104]) define $\rho\colon B(\mathscr{H}) \to C$ by $\rho(T) = \mathrm{Lim}(Tx_n, x_n)$. Then ρ is a positive linear functional of norm one. Let $\pi_\rho\colon B(\mathscr{H}) \to B(\mathscr{H}_\rho)$ be the representation with cyclic vector x_ρ constructed from ρ by the GNS construction, see [10, 4.5.2]. Since (A, W) is strongly controllable, $I = \sum_{k=0}^n A^k W C_k$ for some $C_k \in B(\mathscr{H})$, and it follows that $(\pi_\rho(A), \pi_\rho(W))$ is strongly controllable. It is easily seen (from Roberts' iterative algorithm, for instance) that $s(\pi_\rho(A)) = \pi_\rho(s(A))$. Then by Theorem 3,

$$\ker(\pi_\rho(L)) \subset \bigcap_{j=0}^{\infty} \ker(\pi_\rho(W)\, \pi_\rho(A^{*j})) = \{0\}.$$

But $\rho(L)=0$, so $\pi_\rho(L)\,x_\rho=0$; and this contradicts $\ker(\pi_\rho(L))=\{0\}$. Hence L is invertible. Now $\mathscr{H}=s^-(H)(\mathscr{H})\oplus s^+(H)(\mathscr{H})$ is an orthogonal decomposition of $\mathscr{H}$. With respect to this decomposition $s(A)\,H+Hs(A)^*=L$ becomes

$$\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}+\begin{pmatrix} H_1 & 0 \\ 0 & H_2 \end{pmatrix}\begin{pmatrix} Z_1^* & Z_3^* \\ Z_2^* & Z_4^* \end{pmatrix}=\begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix},$$

where $s(A)=\left(\begin{smallmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{smallmatrix}\right)$, $H=\left(\begin{smallmatrix} H_1 & 0 \\ 0 & H_2 \end{smallmatrix}\right)$. Then

$$Z_1H_1+H_1Z_1^*=L_1$$

$$Z_4H_2+H_2Z_4^*=L_2,$$

where L_1 and L_2 are positive and invertible, H_1 is negative and invertible, H_2 is positive and invertible. It follows from the infinite-dimensional version of Lyapunov's theorem, see [18, Theorem 4], that $\sigma(Z_1)$ is contained in the open left half-plane and $\sigma(Z_4)$ is contained in the open right half-plane. In particular, Z_1-I and Z_4+I are invertible. We can now show that $s^+(H)$: $s^+(A)(\mathscr{H})\to s^+(H)(\mathscr{H})$ is one-to-one. Let $u\in s^+(A)(\mathscr{H})$. Then with respect to the decomposition of $\mathscr{H}$ as $s^-(H)(\mathscr{H})\oplus s^+(H)(\mathscr{H})$, $u=(x, y)$. Since $s^+(A)=\frac{1}{2}(s(A)+I)$ and $s^+(A)u=u$, it follows that $s(A)u=u$. Thus

$$\begin{pmatrix} Z_1 & Z_2 \\ Z_3 & Z_4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix},$$

so $Z_1x+Z_2y=x$, or $x=(I-Z_1)^{-1}Z_2y$. But if $s^+(H)u=0$, then $y=0$, so $x=0$ and $u=0$. So $s^+(H)$ is one-to-one when restricted to $s^+(A)(\mathscr{H})$.

We now show that $s^+(H)$ maps $s^+(A)(\mathscr{H})$ onto $s^+(H)(\mathscr{H})$. For $y\in s^+(H)(\mathscr{H})$, let

$$x=(I-Z_1)^{-1}Z_2y$$

and let $u\in\mathscr{H}=s^-(H)(\mathscr{H})\oplus s^+(H)(\mathscr{H})$ be given by $u=(x, y)$. Then $s^+(H)u=y$, so we need only show that $s^+(A)u=u$ or $s(A)u=u$. An easy calculation shows that $s(A)u=u$ if and only if

$$(Z_3(I-Z_1)^{-1}Z_2+Z_4-I)\,y=0, \tag{2}$$

where I is the identity on $s^+(H)(\mathscr{H})$. Since $s(A)^2=I$, we have that

$$0=\begin{pmatrix} I & 0 \\ Z_3(I-Z_1)^{-1} & I \end{pmatrix}(s(A)-I)(s(A)+I)$$

$$=\begin{pmatrix} Z_1-I & Z_2 \\ 0 & X \end{pmatrix}\begin{pmatrix} Z_1+I & Z_2 \\ Z_3 & Z_4+I \end{pmatrix},$$

where $X = Z_3(I - Z_1)^{-1} Z_2 + Z_4 - I$. It follows that $X = 0$ and Eq. (2) is satisfied. Hence $u \in s^+(A)(\mathscr{H})$ and $s^+(H)u = y$, and $s^+(H)$ maps $s^+(A)(\mathscr{H})$ one-to-one and onto $s^+(H)(\mathscr{H})$. Since $s^-(A) = s^+(-A)$ (as is easily seen by using the iterative algorithm to compute $s(A)$), applying what we just proved to $-A$ and $-H$ yields that $s^-(H)$ maps $s^-(A)(\mathscr{H})$ one-to-one and onto $s^-(H)(\mathscr{H})$. Q.E.D.

As a corollary of Theorem 4, we have the following infinite-dimensional version of Sylvester's law of inertia. In [5, Theorem 6.1] it is proved, without the assumption of H being invertible, that there are continuous one-to-one linear maps from $s^+(S^*HS)(\mathscr{H})$ into $s^+(H)(\mathscr{H})$ and from $s^+(H)(\mathscr{H})$ into $s^+(S^*HS)(\mathscr{H})$. (In that case, $s^+(H)$ is the spectral projection of H associated with that part of $\sigma(H)$ in the open right half-plane.)

COROLLARY 5. *Let H, $S \in B(\mathscr{H})$ with H Hermitian and invertible and S invertible. Then the linear transformations*

$$s^{\pm}(H)\, S^{*-1}: s^{\pm}(S^*HS)(\mathscr{H}) \longrightarrow s^{\pm}(H)(\mathscr{H})$$

are both one-to-one and onto.

Proof. Let $A = HSS^*$. Let $0 < \delta$ be such that $\delta 1 \leqslant SS^*$. Then $AH + HA^* = 2HSS^*H \geqslant 2\delta^2 H^2$, so $AH + HA^*$ is positive and invertible. It follows from Theorem 4 that

$$s^+(H): s^+(A)(\mathscr{H}) \to s^+(H)(\mathscr{H})$$

is one-to-one and onto. The iterative algorithm for computing $s(A)$ shows that if T is invertible, then $s^+(TAT^{-1}) = Ts^+(A)\, T^{-1}$. Thus

$$s^+(S^*HS) = s^+(S^*AS^{*-1}) = S^*s^+(HSS^*)\, S^{*-1}.$$

It follows that

$$S^{*-1}: s^+(S^*HS)(\mathscr{H}) \to s^+(A)(\mathscr{H})$$

is one-to-one and onto. Q.E.D.

Less detailed versions of the following theorem were proved for finite dimensions in [12, Theorem 1] and for infinite dimensions in [18, Theorem 6; 3, Theorem 3].

THEOREM 6. *Let $A \in B(\mathscr{H})$ with $\delta(A) = 0$. Let $W \in B(\mathscr{H})$ be positive with $W = s^+(A)\, Ws^+(A)^* + s^-(A)\, Ws^-(A)^*$. Then there is an Hermitian $H \in B(\mathscr{H})$ with $AH + HA^* = W$.*

Proof. As in the proof of Lemma 2, let $S_{\pm}$ be Cauchy domains with boundaries $C_{\pm}$ such that

$$\{z \in \sigma(A): \pm \operatorname{Re}(z) > 0\} \subset S_{\pm} \subset \{z: \pm \operatorname{Re}(z) > \varepsilon\},$$

where $\varepsilon > 0$. Let $s^{\pm} = s^{\pm}(A)$ and

$$H_{\pm} = \pm \int_{\mp\infty}^{0} e^{tA} s^{\pm} W(s^{\pm})^{*} e^{tA^{*}} \, dt.$$

It was shown in the proof of Lemma 2 that these improper integrals are norm convergent. Also

$$\begin{aligned} AH_{\pm} + H_{\pm} A^{*} &= \pm \int_{\mp\infty}^{0} \frac{d}{dt} (e^{tA} s^{\pm} W(s^{\pm})^{*} e^{tA^{*}}) \, dt \\ &= \pm s^{\pm} W(s^{\pm})^{*}. \end{aligned}$$

Thus $A(H_{+} - H_{-}) + (H_{+} - H_{-}) A^{*} = W.$ Q.E.D.

We now consider a form of a converse of Theorem 4. The finite-dimensional version is in [20, Theorem 2]. Let (c_{ij}) for $i, j = 0, 1, \ldots, n-1$, be an $n \times n$ Hermitian complex matrix. Let

$$f(\lambda) = \sum_{i,j=0}^{n-1} c_{ij} \bar{\lambda}^{i} \lambda^{j},$$

and for A and H in $B(\mathscr{H})$ with H Hermitian let

$$f_{H}(A) = \sum_{i,j=0}^{n-1} c_{ij} A^{*i} H A^{j}.$$

Common examples of such $f_{H}(A)$ are $HA + A^{*}H$ and $H - A^{*}HA$. We begin with an easy lemma. For A in $B(\mathscr{H})$ the approximate point spectrum of A, $a(A)$, is the set of complex numbers λ such that there is a sequence of unit vectors x_n in $\mathscr{H}$ such that $(A - \lambda I) x_n$ converges to zero in norm.

LEMMA 7. *Let H, A, and f be as above with $W = f_{H}(A)$. If x_k is a sequence of unit vectors in $\mathscr{H}$ with $(A - \lambda I) x_k$ converging to zero in norm as k goes to infinity, then $([W - f(\lambda)H] x_k, x_k)$ converges to zero as k goes to infinity.*

Proof. By the definitions, we have that $W - f(\lambda)H$ is a finite linear combination of terms of the form $A^{*p}HA^{q} - \bar{\lambda}^{p}\lambda^{q}H$. But

$$\begin{aligned} &([A^{*p}HA^{q} - \bar{\lambda}^{p}\lambda^{q}H]x_k, x_k) \\ &\qquad = (HA^{q}x_k, (A^{p} - \lambda^{p}I)x_k) + (H(A^{q} - \lambda^{q}I)x_k, \lambda^{p}x_k) \end{aligned}$$

which coverges to zero as k goes to infinity. Q.E.D.

COROLLARY 8. *Let H be positive and invertible and let $\lambda \in a(A)$. If $W = f_H(A)$ is positive and invertible, positive, or zero, respectively, then $f(\lambda)$ is >0, $\geqslant 0$, or $=0$, respectively.*

Proof. In the notation of Lemma 7, we have that

$$(Wx_k, x_k) - f(\lambda)(Hx_k, x_k)$$

converges to zero. By hypothesis, $(Hx_k, x_k) \geqslant h$ for some $h > 0$. The cases $W = 0$ and $W \geqslant 0$ are then clear. If W is positive and invertible, note that $(Wx_k, x_k) \geqslant w$ for some $w > 0$. Q.E.D.

THEOREM 9. *If $f_H(A) = W$ with W positive and H positive and invertible, then $f(\lambda) > 0$ for all λ in $a(A)$ if and only if the pair (A^*, W) is strongly controllable.*

Proof. By Corollary 8, $f(\lambda) \geqslant 0$ for all λ in $a(A)$. For λ in $a(A)$ choose a sequence of unit vectors x_k with $(A - \lambda I)\, x_k$ converging to zero in norm. If $f(\lambda) = 0$, then Lemma 7 implies that (Wx_k, x_k) converges to zero. Hence Wx_k converges to zero in norm, as does

$$WA^j x_k = W(A^j - \lambda^j I)\, x_k + \lambda^j W x_k$$

for $j = 0, 1, 2, \ldots$. If (A^*, W) were strongly controllable, then there would exist $C_0, C_1, C_2, \ldots, C_m$ in $B(\mathscr{H})$ such that

$$I = \sum_{j=0}^{m} A^{*j} W C_j = \sum_{j=0}^{m} C_j^* W A^j.$$

It would then follow that x_k converges to zero, a contradiction.

Conversely, if (A^*, W) is not strongly controllable there is a Hilbert space $\mathscr{H}_\pi$ and a $*$-representation π: $B(\mathscr{H}) \to B(\mathscr{H}_\pi)$ such that $(\pi(A^*), \pi(W))$ is not weakly controllable, see [2, Theorem 13]. The subspace

$$L = [\pi(A^*)^j\, \pi(W)x \colon j \geqslant 0, x \in \mathscr{H}_\pi],$$

where $[\cdot]$ means the norm-closed linear span of the set inside the brackets, is then properly contained in $\mathscr{H}_\pi$. Since L is an invariant subspace for $\pi(A^*)$, the matrix decomposition of $\pi(A^*)$ and $\pi(W)$ with respect to $\mathscr{H}_\pi = L \oplus L^\perp$ is

$$\pi(A^*) = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

$$\pi(W) = \begin{pmatrix} W_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Let λ be any complex number in $a(A_{22}^*)$ and let x_n be a sequence of unit vectors in $L^\perp$ such that $(A_{22}^* - \lambda I)\, x_n$ converges to zero in norm. Then

$$(\pi(A) - \lambda I)\begin{pmatrix} 0 \\ x_n \end{pmatrix} = \begin{pmatrix} A_{11}^* - \lambda & 0 \\ A_{12}^* & A_{22}^* - \lambda \end{pmatrix}\begin{pmatrix} 0 \\ x_n \end{pmatrix}$$
$$= \begin{pmatrix} 0 \\ (A_{22}^* - \lambda)\, x_n \end{pmatrix}$$

converges to zero and

$$\left(\pi(W)\begin{pmatrix} 0 \\ x_n \end{pmatrix}, \begin{pmatrix} 0 \\ x_n \end{pmatrix}\right) = 0.$$

Thus $\lambda \in a(\pi(A))$ and (by [1, Corollary 5], for instance) $a(\pi(A)) \subseteq a(A)$. Let Lim be any generalized limit on the space of bounded complex sequences. Define a positive linear map $\rho: B(\mathscr{H}_\pi) \to C$ by

$$\rho(T) = \text{Lim}\left(T\begin{pmatrix} 0 \\ x_n \end{pmatrix}, \begin{pmatrix} 0 \\ x_n \end{pmatrix}\right).$$

Then $\rho(\pi(W)) = 0$ and for all $T \in B(\mathscr{H}_\pi)$, $\rho(T\pi(A)) = \rho(T)\lambda$. Then

$$0 = \rho(\pi(W)) = \rho(\pi(f_H(A))) = \sum c_{ij}\rho(\pi(A^{*i}HA^j))$$
$$= \left(\sum c_{ij}\bar{\lambda}^i\lambda^j\right)\rho(\pi(H)).$$

So $f(\lambda)\,\rho(\pi(H)) = 0$ and $f(\lambda) = 0$ since $\pi(H)$ is positive and invertible. Since $\lambda \in a(A)$, this completes the proof of the theorem. Q.E.D.

Two special cases of Theorem 9 seem worth singling out. See [20, Theorem 4] for the finite-dimensional case.

COROLLARY 10. *Let A, H, $W \in B(\mathscr{H})$ with W positive and invertible. Then*

(a) *If $A^*H + HA = W$, then $\sigma(A)$ is contained in the open right half-plane if and only if (A^*, W) is strongly controllable.*

(b) *If $H - A^*HA = W$, then $\sigma(A)$ is contained in the open unit disk if and only if (A^*, W) is strongly controllable.*

Let A be an $n \times n$ complex matrix and B an $n \times m$ complex matrix. Hautus has proved [9] that the pair (A, B) is controllable if and only if the $n \times (n+m)$ matrix $[A - \lambda I, B]$ has rank n for every eigenvalue λ of A. The methods used to prove Theorem 8 can also be used to prove the following infinite-dimensional version of this theorem.

THEOREM 11. *Let A, $B \in B(\mathscr{H})$. Then (A, B) is strongly controllable if and only if $[A - \bar{\lambda} I, B]$ maps $\mathscr{H} \oplus \mathscr{H}$ onto $\mathscr{H}$ for every $\lambda \in a(A^*)$.*

Proof. If (A, B) is not strongly controllable, then there is a Hilbert space $\mathscr{H}_\pi$ and a *-representation $\pi: B(\mathscr{H}) \to B(\mathscr{H}_\pi)$ such that $(\pi(A), \pi(B))$ is not weakly controllable; see [2; Theorem 13]. The subspace

$$L = [\pi(A^j)\,\pi(B)x: j \geqslant 0, x \in \mathscr{H}_\pi],$$

where $[\cdot]$ means the norm closure of the linear span of the set inside the brackets, is then properly containing in $\mathscr{H}_\pi$. The matrix decomposition of $\pi(A)$ and $\pi(B)$ with respect to $\mathscr{H}_\pi = L \oplus L^\perp$ is

$$\pi(A) = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix},$$

$$\pi(B) = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix}.$$

Let $\lambda \in a(A_{22}^*)$ and choose unit vectors $x_n \in L^\perp$ with $(A_{22}^* - \lambda I)\,x_n$ converging to zero in norm. Let

$$u_n = \begin{pmatrix} 0 \\ x_n \end{pmatrix}.$$

Then $(\pi(A)^* - \lambda I)\,u_n$ and $\pi(B^*)\,u_n$ both converge to zero in norm. Thus

$$(\pi(A) - \bar{\lambda} I)(\pi(A^*) - \lambda I) + \pi(B)\,\pi(B^*)$$

is not invertible, so

$$[A - \bar{\lambda} I, B][A - \bar{\lambda} I, B]^* = (A - \bar{\lambda} I)(A^* - \lambda I) + BB^*$$

is not invertible. Hence $[A - \bar{\lambda} I, B]$ is not onto, but $\lambda \in a(\pi(A^*)) \subseteq a(A^*)$.

Conversely, assume (A, B) is strongly controllable and assume

$$I = \sum_{i=0}^{n} A^i B C_i.$$

Let $\lambda \in a(A^*)$. If $[A - \bar{\lambda} I, B]$ were not onto,

$$S = (A - \bar{\lambda} I)(A^* - \lambda I) + BB^*$$

would not be invertible and there would exist a sequence of unit vectors x_n in $\mathscr{H}$ with Sx_n converging to zero, so

$$\begin{aligned} x_n &= \sum_{i=0}^{n} C_i^* B^* A^{i*} x_n \\ &= \sum_{i=0}^{n} C_i^* B^* (A^{i*} - \lambda^i I)\,x_n + \sum_{i=0}^{n} \lambda^i C_i^* B^* x_n \end{aligned}$$

converges to zero, a contradiction. Hence $[A - \bar{\lambda} I, B]$ is onto. Q.E.D.

REFERENCES

1. J. BUNCE, Characters on singly generated C^*-algebras, *Proc. Amer. Math. Soc.* **25** (1970), 297–303.
2. J. BUNCE, Stabilizability of linear systems defined over C^*-algebras, *Math. Systems Theory* **18** (1985), 237–250.
3. B. CAIN, An inertia theory for operators on a Hilbert space, *J. Math. Anal. Appl.* **41** (1973), 97–114.
4. B. CAIN, The inertial aspects of Stein's condition $H - C^*HC \gg 0$, *Trans. Amer. Math. Soc.* **196** (1974), 79–91.
5. B. CAIN, Inertia theory, *Linear Algebra Appl.* **30** (1980), 211–240.
6. C. T. CHEN, A generalization of the inertia theory, *SIAM J. Appl. Math.* **25** (1973), 158–161.
7. P. FUHRMANN, On weak and strong reachability and controllability of infinite-dimensional linear systems, *J. Optim. Theory Appl.* **9** (1972), 77–87.
8. F. R. GANTMACHER, "The Theory of Matrices, Vols. I, II," Chelsea, New York, 1959.
9. M. L. J. HAUTUS, Controllability and observability conditions for linear autonomous systems, *Nederl. Akad. Wetensch. Proc. Ser. A* **72** (1969), 443–448.
10. R. KADISON AND J. RINGROSE, "Fundamentals of the Theory of Operator Algebras, Vol. I," Academic Press, New York, 1983.
11. M. NEWMAN, Lyapunov revisited, *Linear Algebra Appl.* **52** (1983), 525–528.
12. A. OSTROWSKI AND H. SCHNEIDER, Some theorems on the inertia of general matrices, *J. Math. Anal. Appl.* **4** (1962), 72–84.
13. H. Radjavi and P. Rosenthal, "Invariant Subspaces," Springer-Verlag, New York, 1973.
14. J. D. ROBERTS, Linear model reduction and solution of the algebraic Riccati equation by the use of the sign function, *Internat. J. Control* **32** (1980), 677–687.
15. A. SOUROUR, On strong controllability of infinite-dimensional linear systems, *J. Math. Anal. Appl.* **87** (1982), 460–462.
16. H. STETKAER, On positive semidefinite solutions of the operator Lyapunov equation, *J. Math. Anal. Appl.* **69** (1979), 153–170.
17. O. TAUSSKY, A generalization of a theorem of Lyapunov, *J. Soc. Indus. Appl. Math.* **9** (1961), 640–643.
18. J. P. WILLIAMS, Similarity and the numerical range, *J. Math. Anal. Appl.* **26** (1969), 307–314.
19. H. WIMMER, Inertia theorems for matrices, controllability, and linear vibrations, *Linear Algebra Appl.* **8** (1974), 337–343.
20. H. WIMMER, Generalizations of theorems of Lyapunov and Stain, *Linear Algebra Appl.* **10** (1975), 139–146.
21. K. YOSIDA, "Functional Analysis," Springer-Verlag, New York, 1968.